Dirichlet characters and field extensions

Diaa Al Saleh

Definition 1 A Dirichlet character modulo n is a character of $(\mathbb{Z}/n\mathbb{Z})^{\times}$. In other words a multiplicative homomorphism

$$\chi : (\mathbb{Z}/n\mathbb{Z})^{\times} \to S^1$$

We call n the modulus of χ

Example. Let i be the usual complex number. Define $(\mathbb{Z}/5\mathbb{Z})^{\times} \to S^1$ by $\chi(1) = 1, \chi(2) = i, \chi(3) = -i, \chi(4) = -1$

Definition 2 The *conductor* f_χ is the minimal modulus for a Dirichlet character, i.e, χ is not induced by any Dirichlet character of modulus smaller than f_χ. A Dirichlet character with modulus f_χ is called *primitive*.

Now given a fixed positive integer n, the Dirichlet characters having conductors dividing n form a finite group. If we let ζ_n be a primitive n-th root of unity, we can identify $G = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ with $(\mathbb{Z}/n\mathbb{Z})^{\times}$. So a Dirichlet character can be thought as a character on G.

Definition 3 Let χ be a character of G. The fixed field K associated to χ is the fixed field of the kernel of χ

Example.

Let $\chi : G = \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \to S^1$ by $\chi(\sigma_1) = 1, \chi(\sigma_3) = -1, \chi(\sigma_5) = 1, \chi(\sigma_7) = -1$, where $\sigma_j : \zeta_{12} \to \zeta_{12}^j$. Then $\ker \chi = \{\sigma_1, \sigma_5\}$.

It's elementary to see that the fixed field is $\mathbb{Q}(i)$.

Instead if we set $\chi(\sigma_7) = 1$ then the fixed field is real since σ_7 is complex conjugation.

$$\begin{tikzcd}
 & \mathbb{Q}(\zeta_8) \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
\mathbb{Q}(i) \arrow[dr, no head] & \mathbb{Q}\left(\sqrt{2}\right) \arrow[d, no head] & \mathbb{Q}\left(\sqrt{-2}\right) \arrow[dl, no head] \\
 & \mathbb{Q} &
\end{tikzcd}$$

From this we can see that the fixed field is $\mathbb{Q}\left(\sqrt{2}\right)$